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ON ABSENCE OF ASYMPTOTIC STABILITY WITH RESPECT TO A PART OF THE VARIABLES

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We use a generalization of the Liouville formula to state a necessary condition under which the zero solution of a system of nonlinear differential equations has no attraction property with respect to any of the variables. In particular, from the basic theorem it follows that the stable unperturbed motion of a general (nonstationary) Hamiltonian system cannot be attractive with respect to any of the generalized coordinates and impulses. Properties of stability of the equilibrium position of a mathematical pendulum of variable length are investigated as an example.

1. Let the following system of differential equations of perturbed motion be given:

$$\mathbf{x}^\cdot = \mathbf{X}(t, \mathbf{x}) \quad (\mathbf{X}(t, 0) \equiv 0) \tag{1.1}$$

$$\mathbf{x} = (x_1, x_2, \ldots, x_n)^* \in R^n, \quad \|\mathbf{x}\| = (x_1^2 + x_2^2 + \ldots + x_n^2)^{1/2}$$

The vector function $\mathbf{X}(t, \mathbf{x})$ is defined and continuous together with its first order partial derivatives in x_i $(i = 1, 2, \ldots, n)$ on the set $\Gamma = \{(t, \mathbf{x}) : t \geqslant 0, \|\mathbf{x}\| < H\}$ $(0 < H \leqslant \infty)$ and the solutions $\mathbf{x}(t; t_0, \mathbf{x}_0)$ are defined for all $t \geqslant t_0$ provided that the initial values $\mathbf{x}_0 = \mathbf{x}(t_0; t_0, \mathbf{x}_0)$ are sufficiently small in the norm.

Definition. The unperturbed motion $\mathbf{x} = 0$ shall be called attractive with respect to the variable x_j $(1 \leqslant j \leqslant n)$, if for every t_0 there exists $\delta(t_0) > 0$ such that $\|\mathbf{x}_0\| < \delta$ implies

$$\lim_{t\to\infty} x_j(t; t_0, \mathbf{x}_0) = 0 \tag{1.2}$$

An unperturbed motion attractive with respect to all variables $x_1, x_2, \ldots, x_n$ shall simply be called an attractive one.

Using the above terminology we can say that an unperturbed motion is asymptotically x_j-stable [1] if it is x_j-stable and attractive with respect to x_j. Below we shall use the following notation:

$$S(r) = \{\mathbf{x} : \|\mathbf{x}\| < r\} \quad (0 < r \in R)$$

$$x(t; t_0, F) = \{\mathbf{x}(t; t_0, \mathbf{x}_0) : \mathbf{x}_0 \in F\} \quad (F \subset R^n)$$

2. The mapping $S(r) \to R^n$ defined by the formula $\mathbf{x}_0 \to \mathbf{x}(t; t_0, \mathbf{x}_0)$ is, for any fixed t, t_0 $(0 \leqslant t_0 \leqslant t)$, a diffeomorphism the Jacobian of which satisfies the following differential equation [2, 3]:

$$\frac{\partial}{\partial t} J(\mathbf{x}_0; t, t_0) = \sum_{i=1}^{n} \left(\frac{\partial X_i(t, \mathbf{x})}{\partial x_i}\right)_{\mathbf{x}=\mathbf{x}(t;\, t_0,\, \mathbf{x}_0)} J(\mathbf{x}_0; t, t_0)$$

when $t \geqslant t_0$. This yields the relation

$$J(\mathbf{x}_0;\, t,\, t_0) = \exp\left[\int_{t_0}^{t} \left(\sum_{i=1}^{n} \frac{\partial X_i(s,\mathbf{x})}{\partial x_i}\right)_{\mathbf{x}=\mathbf{x}(s;\, t_0,\, \mathbf{x}_0)} ds\right] \tag{2.1}$$

which represents a generalization of the Liouville formula for the systems of nonlinear equations.

Theorem. Assume that a neighborhood of the coordinate origin exists such that all solutions of the system (1.1) originating in this neighborhood are uniformly bounded, i.e. numbers $l > 0$ and $L > 0$ can be found such that

$$x\,(t;\ 0,\ S\,(l)) \subset S\,(L) \quad (t \geqslant 0) \tag{2.2}$$

If

$$\limsup_{t\to\infty} \int_0^t \min\left\{\sum_{i=1}^{n} \frac{\partial X_i(s,\mathbf{x})}{\partial x_i} : \|\mathbf{x}\| \leqslant L\right\} ds > -\infty \tag{2.3}$$

then the zero solution of the system (1.1) has no attractive property with respect to any of the variables $x_1,\ x_2,\ \ldots,\ x_n$ or, more accurately, the Lebesgue measures $\mu\,[E_i]$ of the sets

$$E_i = \{\mathbf{x}_0 : \|\mathbf{x}_0\| < l,\quad \lim_{t\to\infty} x_i(t;\, 0,\, x_0) = 0\}$$

$(i = 1,\ 2,\ \ldots,\ n)$ are equal to zero.

Proof. By virtue of the condition (2.3) and relation (2.1) a sequence $0 \leqslant t_1 \leqslant \ldots \leqslant t_k \leqslant \ldots$ and a constant C exist such that $t_k \to \infty$ when $k \to \infty$ and the inequality

$$\mu\,[x\,(t_k;\, 0,\, F)] = \int \ldots \int_{x(t_k;\ 0,\ F)} dx_1 \ldots dx_n = \tag{2.4}$$

$$\int \ldots \int_F J\,(\mathbf{x}_0;\, t_k,\, 0)\, dx_{01} \ldots dx_{0n} \geqslant \exp\,[C]\,\mu\,[F] \quad (k = 1, 2, \ldots)$$

holds for any open measurable set $F \subset S\,(l)$. The sets

$$H_m^k = \left\{\mathbf{x}_0 : \|\mathbf{x}_0\| < l,\ |x_i(t;\, 0,\, \mathbf{x}_0)| < \frac{1}{k} \text{ при } m \leqslant t \leqslant m+1\right\}$$
$$(m,\, k = 1, 2, \ldots)$$

are open for any fixed i $(1 \leqslant i \leqslant n)$ (see [2]), therefore the set

$$E_i = \bigcap_{k=1}^{\infty} \Big(\bigcup_{j=1}^{\infty} \bigcap_{m=j}^{\infty} H_m^k\Big)$$

is Lebesgue measurable.

Let us assume that the theorem is incorrect, i.e. that there exists j $(1 \leqslant j \leqslant n)$ such that $\mu\,[E_j] > 0$. Then using a theorem due to D. F. Egorov [4] we can find a measurable set $E^* \subset E_j$ the measure of which satisfies the inequality $\mu\,[E^*] > \mu\,[E_j]/2$ and on which $x_j\,(t;\ 0,\ \mathbf{x}_0) \to 0$ uniformly in $\mathbf{x}_0$ when $t \to \infty$, i.e. for any $\varepsilon > 0$, $T\,(\varepsilon)$ can be found such that $t > T\,(\varepsilon)$ and $\mathbf{x}_0 \in E^*$ implies the inclusion

$$\mathbf{x}\,(t;\ 0,\ \mathbf{x}_0) \in M\,(\varepsilon) \tag{2.5}$$
$$M\,(\varepsilon) = S\,(L) \cap \{\mathrm{y}:\ |y_j| < \varepsilon\}$$

Since $t_k \to \infty$, a natural number $k\,(\varepsilon)$ can be found such that $t_{k(\varepsilon)} > T\,(\varepsilon)$. Let us introduce the notation

$$F\,(\varepsilon) = x\,(0;\ t_{k(\varepsilon)},\ M\,(\varepsilon)) \cap S\,(l)$$

By virtue of (2.5) we have $E^* \subset F\,(\varepsilon)$, therefore $\mu\,[F\,(\varepsilon)] \geqslant \mu\,[E_j]\,/\,2$. Using now (2.4), we obtain the following estimate:

$$\mu\,[M\,(\varepsilon)] \geqslant \mu\,[x\,(t_{k(\varepsilon)};\ 0,\ F\,(\varepsilon))] \geqslant \exp\,[C]\ \mu\,[F\,(\varepsilon)] \geqslant \exp\,[C]\ \mu\,[E_j]\,/\,2 > 0 \tag{2.6}$$

On the other hand, we have the obvious relation

$$\lim_{\varepsilon\to 0+}\mu\,[M\,(\varepsilon)] = 0$$

which contradicts the estimate (2.6) thus proving the theorem.

Corollary 1. If the zero solution of the system (1.1) is stable and the inequality (2.3) holds for sufficiently small $L > 0$, then the zero solution of (1.1) has no attraction property towards any of the variables x_i; to express it more accurately, $\mu\,[E_i] = 0$ $(i = 1,\ 2,\ \ldots,\ n)$.

Now let the following arbitrary (nonconservative) Hamiltonian system be given

$$\dot{q}_i = \frac{\partial H\,(t,\ \mathbf{q},\ \mathbf{p})}{\partial p_i},\quad \dot{p}_i = -\frac{\partial H\,(t,\ \mathbf{q},\ \mathbf{p})}{\partial q_i}\qquad (i = 1, 2, \ldots, n) \tag{2.7}$$

and let us assume that the Hamiltonian function $H\,(t,\ \mathbf{q},\ \mathbf{p}): [0,\ \infty) \times R^n \times R^n \to R$ is continuous together with its second partial derivatives in q_i and p_j. Let the system (2.7) have a solution $\mathbf{q} = \mathbf{p} = 0$ which we shall call the position of equilibrium.

Corollary 2. If a neighborhood of the position of equilibrium $\mathbf{q} = \mathbf{p} = 0$ of the system (2.7) exists in the $2n$-dimensional space of the variables $\mathbf{q},\ \mathbf{p}$ such that all solutions originating in this neighborhood are uniformly bounded, i.e. numbers $l > 0$ and $L > 0$ exist such that $\|\mathbf{q}_0\|^2 + \|\mathbf{p}_0\|^2 \leqslant l^2$ implies the inequality

$$\|\mathbf{q}\,(t;\,0,\,\mathbf{q}_0,\,\mathbf{p}_0)\|^2 + \|\mathbf{p}\,(t;\,0,\,\mathbf{q}_0,\,\mathbf{p}_0)\|^2 \leqslant L^2 \tag{2.8}$$

for all $t \geqslant 0$ (in particular when this position of equilibrium $\mathbf{q} = \mathbf{p} = 0$ is stable), then the position of equilibrium has no attraction property with respect to any of the variables q_i, p_i, or more accurately, the Lebesgue measures of the sets

$$\begin{pmatrix} Q_i \\ P_i \end{pmatrix} = \left\{(\mathbf{q}_0,\,\mathbf{p}_0) \in R^{2n} : \|\mathbf{q}_0\|^2 + \|\mathbf{p}_0\|^2 < l^2,\ \lim_{t\to\infty} \begin{pmatrix} q_i \\ p_i \end{pmatrix} (t;\ 0,\ \mathbf{q}_0,\ \mathbf{p}_0) = 0\right\}$$

$(i = 1,\ 2,\ \ldots,\ n)$ are all equal to zero.

Note. The equation

$$\ddot{x} + a\,(t)x = 0 \quad (t \geqslant 0,\ x \in R) \tag{2.9}$$

shows that the condition (2.8) in Corollary 2 is essential. Setting $q = x$ and $p = \dot{x}$, we can write (2.9) in the form of the Hamiltonian system (2.7) with the function $H\,(t,\ q,\ p) = (a\,(t)\,q^2 + p^2)\,/\,2$. The solution $x = \dot{x} = 0$ of (2.9) cannot be attractive irrespective of what the function $a\,(t)$ is. On the other hand, the problem of the conditions under which all solutions of (2.9) tend to zero as $t \to \infty$, or setting it differently, when the solution $x = \dot{x} = 0$ of (2.9) is attractive (in the whole) with respect to the coordinate x, has been a subject of study for a long time. A number of conditions guaranteeing this property are known (see Sect. 5.5 of [5]). It follows that the rejection of the condition (2.8) invalidates Corollary 2.

3. As an example, we consider the motion of a pendulum consisting of a material point suspended by a thread the length of which varies in accordance with an arbitrarily stated law $l = l(t)$ $(l(t) \geqslant l_0 > 0)$. We denote by θ the angle formed by the thread with the vertical. In this case the Lagrange equation has the form

$$(l^2(t)\theta^{\cdot})^{\cdot} + gl(t)\sin\theta = 0 \quad (-\pi/2 < \theta < \pi/2) \tag{3.1}$$

Consider the "normalized energy"

$$V = V(t, \theta, \theta^{\cdot}) = \frac{l(t)}{g}(\theta^{\cdot})^2 + 2(1 - \cos\theta) \tag{3.2}$$

By virtue of Eq. (3.1) we can write the following estimate for the derivative $V^{\cdot}$:

$$V^{\cdot}(t, \theta, \theta^{\cdot}) = -\frac{3}{g} l^{\cdot}(t)(\theta^{\cdot})^2 \leqslant \left[\frac{3l^{\cdot}(t)}{l(t)} l(t)\right]_- V(t, \theta, \theta^{\cdot}) \tag{3.3}$$

Assume that

$$L = \int_0^\infty [(\ln l(t))^{\cdot}]_- \, dt < \infty \tag{3.4}$$

Then any one solution $\theta(t)$ of (3.1) satisfies the inequality

$$v(t) = V(t, \theta(t), \theta^{\cdot}(t)) \leqslant v(t_0)\exp[3L] \tag{3.5}$$

Since $(1 - \cos\theta)/\theta^2 \to 1/2$ as $\theta \to 0$, for any $\sigma > 0$ and $t_0 \geqslant 0$, there exist $k > 0$ and $K(t_0)$ such that

$$V(t, \theta, \theta^{\cdot}) \geqslant k(\theta^2 + (\theta^{\cdot})^2)$$
$$V(t_0, \theta, \theta^{\cdot}) \leqslant K(t_0)(\theta^2 + (\theta^{\cdot})^2)$$
$$(t \geqslant 0, \theta^{\cdot} \in R, \quad 0 \leqslant |\theta| < \pi/2 - \sigma)$$

If $\varepsilon > 0$ and

$$\theta_0^2 + (\theta_0^{\cdot})^2 < \varepsilon \frac{k}{K(t_0)\exp[3L]}$$

then by virtue of (3.5) we have the inequality

$$[\theta(t; t_0, \theta_0, \theta_0^{\cdot})]^2 + [\theta^{\cdot}(t; t_0, \theta_0, \theta_0^{\cdot})]^2 < \varepsilon \quad (t \geqslant t_0)$$

i.e. the condition (3.4) entails the stability of the unperturbed motion $\theta = \theta^{\cdot} = 0$.

We note that the condition (3.4) obviously holds when the function $l(t)$ $(l(t) \geqslant l_0 > 0)$ *increases or decreases, at sufficiently large values of* t. If an unbounded sequence of the time instances $r_1 < s_1 < \ldots < r_k < s_k < \ldots$ is such that the function $l(t)$ decreases on the intervals $[r_k, s_k]$ and increases on the intervals $[s_k, r_{k+1}]$ $(k = 1, 2, \ldots)$, then the condition (3.4) is equivalent to the inequality

$$\prod_{k=1}^{\infty} (l(r_k)/l(s_k)) < \infty$$

Let us find for what function $l(t)$ the unperturbed motion $\theta = \theta^{\cdot} = 0$ is attractive with respect to the angle θ. A simple computation shows that when the system is equivalent to (3.1), the condition (2.3) is equivalent to the inequality $\liminf_{t\to\infty} l(t) < \infty$. Using Corollary 1 we find that if the function $l(t)$ is bounded and satisfies the condition (3.4), then the unperturbed motion $\theta = \theta^{\cdot} = 0$ cannot be attractive (and hence asymptotically stable) neither with respect to the angle θ, nor with respect to the angular velocity $\theta^{\cdot}$.

Let us consider the case when the function $l(t)$ is unbounded; in particular let us assume that

$$l(t) = l_0 + ct^{\alpha} \quad (0 < l_0, c, \alpha = \text{const}) \tag{3.6}$$

We shall show that when $0 < \alpha \leqslant 2$, the unperturbed motion $\theta = \theta^{\cdot} = 0$ is attractive with respect to the angle θ, i.e. all solutions of (3.1) defined on the interval $[t_0, \infty)$ tend to zero as $t \to \infty$.

Assume that the solution $\theta(t)$ is nonoscillatory. In this case it varies monotonously at sufficiently large values of t and tends to a finite limit ν, since all solutions are bounded. Let us assume that $\nu \neq 0$, e.g. $\nu > 0$. Integrating Eq.(3.1) twice from a sufficiently large value T_0, we obtain the estimate

$$\theta(t) \leqslant \theta(T_0) + l(T_0)\,|\,\theta^{\cdot}(T_0)\,|\int_{T_0}^{t}(l_0 + cs^{\alpha})^{-2}\,ds - c_1 \sin\nu \int_{T_0}^{t} s^{1-\alpha}ds \to -\infty \quad (t \to \infty)$$

which contradicts the fact that the function $\theta(t)$ is bounded, and hence $\nu = 0$.

Let us now assume that the solution $\theta(t)$ oscillates, i.e. a sequence $t_1 < t_2 < \cdots < t_k < \cdots$ exists for which

$$\theta(t_k) = 0 \quad (k = 1, 2, \ldots), \quad \lim_{k\to\infty} t_k = 0 \tag{3.7}$$

Setting

$$p(t) = (l_0 + ct^{\alpha})^2, \quad q(t) = g(l_0 + ct^{\alpha})$$

we consider the following auxiliary Liapunov function (see [6]):

$$W = W(t, \theta, \theta^{\cdot}) = dV + \frac{\pi}{2}\frac{d^{\cdot}}{q}p\theta\theta^{\cdot} - \frac{\pi}{4}\left(\frac{d^{\cdot}}{q}\right)p\theta^2$$

where $d = d(t)$ is a thrice continuously differentiable function on the interval $[0, \infty)$. By virtue of (3.1), the derivative of W has the form

$$W^{\cdot} = \frac{p}{q}(\theta^{\cdot})^2\left[\left(1 + \frac{\pi}{2}\right)d^{\cdot} - d\frac{(pq)^{\cdot}}{pq}\right] - \frac{\pi}{4}\left(\left(\frac{d^{\cdot}}{q}\right)^{\cdot}p\right)^{\cdot}\theta^2 + d^{\cdot}\left[2(1 - \cos\theta) - \frac{\pi}{2}\theta\sin\theta\right] \tag{3.8}$$

Now $l^{\cdot}(t) \geqslant 0$, therefore from (3.3) we see that $v(t) = V(t, \theta(t), \theta^{\cdot}(t)) \searrow \lambda \geqslant 0$ as $t \to \infty$.

It is sufficient to show that $\lambda = 0$. Assume the opposite, i.e. that $\lambda > 0$. Then for every $\varepsilon > 0$ there exists $T(\varepsilon)$ such that

$$\lambda \leqslant v(t) \leqslant (1 + \varepsilon)\lambda \quad (t \geqslant T(\varepsilon)) \tag{3.9}$$

Integrating (3.8) from $T(\varepsilon)$ to t_k ($\geqslant T(\varepsilon)$) and using (3.7), we obtain

$$d(t_k)\,v(t_k) \leqslant O(1) + \int_{T}^{t_k} d^{\cdot}\left[1 + \frac{\pi}{2} - \frac{d}{d^{\cdot}}\frac{(pq)^{\cdot}}{pq}\right]_{+} v\,dt + \frac{\pi}{4}\int_{T}^{t_k}\left[\left(\left(\frac{d^{\cdot}}{q}\right)^{\cdot}p\right)^{\cdot}\right]_{-}\theta^2 dt \quad (k \to \infty) \tag{3.10}$$

Assume now that $\delta(t) = t^{\delta}$, where

$$\delta = \delta(\alpha) = \begin{cases} \min(1/2,\ 5\alpha/\pi), & 0 < \alpha < 2 \\ 3, & \alpha = 2 \end{cases}$$

Then

$$\mu = \lim_{t\to\infty}\frac{d\,(pq)^{\cdot}}{d^{\cdot}\,(pq)} = \lim_{t\to\infty}\frac{3\alpha ct^{\alpha}}{\delta(l_0 + ct^{\alpha})} = \frac{3\alpha}{\delta} > \frac{\pi}{2} \tag{3.11}$$

To obtain the required contradiction from (3.10), we must estimate the integral

$$I = I(t; \alpha, \delta) = \frac{1}{g}\int_{1}^{t}\left[\left(\left(\frac{\delta s^{\delta-1}}{l_0 + cs^{\alpha}}\right)^{\cdot}(l_0 + cs^{\alpha})^2\right)^{\cdot}\right]_{-} ds$$

For $0 < \alpha < 2$ we have

$$I(t;\alpha,\delta) \leqslant \frac{1}{g}\,\frac{\delta c\,[(\delta-1-\alpha)(\delta+\alpha-2)]_-}{\delta+\alpha-2}\,(t^{\delta+\alpha-2}-1) = o(t^{\delta}) \qquad (t\to\infty) \quad (3.12)$$

$$I(t;2,3) \equiv 0$$

Using the relations (3.9)—(3.12), we obtain the estimate

$$\lambda t_k^{\delta} = O(1) + \left[1 - \frac{\mu-\pi/2}{2}\right]_+ (1+\varepsilon)\,\lambda t_k^{\delta} + o(t_k^{\delta}) \qquad (k\to\infty)$$

since the solution $\theta(t)$ is bounded. When $k\to\infty$, the above estimate yields the inequality

$$1 \leqslant \left[1 - \frac{1}{2}\left(\mu - \frac{\pi}{2}\right)\right]_+ (1+\varepsilon)$$

which, by virtue of (3.11), contradicts the assumption that $\varepsilon > 0$ is arbitrary. This proves that when $0 < \alpha \leqslant 2$, all solutions $\theta(t)$ tend to zero as $t\to\infty$.

Let us now consider the case when $\alpha > 2$. Integrating Eq.(3.1) twice, we obtain the following estimate for the solution $\theta(t) = \theta(t;t_0,\theta_0,0)$ $(\theta_0 > 0)$:

$$\theta(t) = \theta_0 - \int_{t_0}^{t} \frac{1}{p(s)} \int_{t_0}^{s} q(\tau)\sin\theta(\tau)\,d\tau\,ds \geqslant \theta_0 - c_2 \int_{t_0}^{\infty} s^{1-\alpha}\,ds \qquad (0 < c_2 = \text{const})$$

From this it follows that for any θ_0 $(0 < |\theta_0| < \pi/2)$ a $t_0 \geqslant 0$ exists such that

$$\liminf_{t\to\infty} |\theta(t;t_0,\theta_0,0)| > 0 \qquad (3.13)$$

and this constitutes a proof of the following assertions:

1) if the pendulum length satisfies the condition (3.4), then the unperturbed motion $\theta = \theta^{\cdot} = 0$ is stable;

2) if the pendulum length $l(t)$ is bounded and satisfies (3.4), then the unperturbed motion cannot be attractive (and hence asymptotically stable) with respect to the angle θ, nor with respect to the angular velocity $\theta^{\cdot}$;

3) if the pendulum length varies according to the rule (3.6), then (a) for $0 < \alpha \leqslant 2$, the unperturbed motion is asymptotically stable with respect to θ and all solutions $\theta(t;t_0,\theta_0,\theta_0^{\cdot})$ of Eq.(3.1) defined on the interval $[t_0,\infty)$ tend to zero as $t\to\infty$, and (b) when $\alpha > 2$ for any θ_0 $(0 < |\theta_0| < \pi/2)$ there exists $t_0 \geqslant 0$ such that the solution $\theta(t;t_0,\theta_0,0)$ possesses the property (3.13).

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